Continuity

4 Marks Questions

1. Find the value of k, so that the function f defined below, is continuous at $x=0$, where

$$f(x)=\begin{cases}\left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}.$$

All India 2014C

Given function is

$$f(x)=\begin{cases}\left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

Also, given $f(x)$ is continuous at $x=0$.

$\therefore \quad (\text{LHL})_{x=0}=(\text{RHL})_{x=0}=f(0) \quad \ldots\text{(i)}$

Now, $\quad \text{LHL}=\lim_{x\to 0^-} f(x)$

$$=\lim_{x\to 0^-}\frac{1-\cos 4x}{8x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos(-4h)}{8h^2}$$

[put $x=0-h=-h$, when $x\to 0$, $h\to 0$] **(1)**

$$=\lim_{h\to 0}\frac{1-\cos 4h}{8h^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 2h}{8h^2}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{\sin^2 2h}{4h^2}=\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2=1 \quad \textbf{(1)}$$

At $x=0$, $f(0)=k$

Now, from Eq. (i), we have

$$\text{LHL}=f(0)$$

$$\Rightarrow \quad 1\cdot 1=k\Rightarrow k=1 \quad \textbf{(1)}$$

Hence, for $k=1$, the given function $f(x)$ is continuous at $x=0$. **(1)**

2. If $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{when } x<0\\ a, & \text{when } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{when } x>0\end{cases}$

and f is continuous at $x=0$, then find the value of a. Delhi 2013C

Given,

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{(\sqrt{16+\sqrt{x}})-4}, & \text{when } x > 0 \end{cases}$$

Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}}{(\sqrt{16+\sqrt{x}})-4} = f(0) \quad \textbf{(1)}$$

$$\Rightarrow \lim_{h\to 0} \frac{1-\cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0} \frac{\sqrt{0+h}}{(\sqrt{16+\sqrt{0+h}}-4)} = f(0)$$

$$\Rightarrow \lim_{h\to 0} \frac{1-\cos 4h}{h^2}$$

$$= \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4} \times \frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4} = a \quad \textbf{(1)}$$

$$\Rightarrow \lim_{h\to 0} \frac{2\sin^2 2h}{h^2} = \lim_{h\to 0} \frac{\sqrt{h}(\sqrt{16+\sqrt{h}}+4)}{(16+\sqrt{h})-16} = a$$

$[\because 1-\cos 2\theta = 2\sin^2\theta$ and $(a+b)(a-b) = a^2-b^2]$

$$\Rightarrow 2\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 \times 4$$

$$= \lim_{h\to 0} \frac{\sqrt{h}(\sqrt{16+\sqrt{h}}+4)}{\sqrt{h}} = a \quad \textbf{(1)}$$

$$\left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$\Rightarrow \quad 2\times(1)^2 \times 4 = \sqrt{16+\sqrt{0}}+4 = a$$

$$\Rightarrow \quad 8 = 4+4 = a$$

$$\therefore \quad a = 8 \quad \textbf{(1)}$$

3. Find the value of k, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at $x=0$. **All India 2013**

Given, $f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$

is continuous at $x = 0$.

Now, $f(0) = \dfrac{2 \cdot 0 + 1}{0 - 1} = \dfrac{1}{-1} = -1$ **(1)**

$\text{LHL} = \lim_{h \to 0} f(0 - h)$

$= \lim_{h \to 0} \dfrac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$

$= \lim_{h \to 0} \dfrac{\sqrt{1-kh} - \sqrt{1+kh} \times (\sqrt{1-kh} + \sqrt{1+kh})}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$ **(1)**

$= \lim_{h \to 0} \dfrac{(1-kh) - (1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$

$[\because (a+b)(a-b) = a^2 - b^2]$

$= \lim_{h \to 0} \dfrac{-2kh}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$ **(1)**

$= \lim_{h \to 0} \dfrac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} = \dfrac{2k}{1+1} = \dfrac{2k}{2} = k$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = \text{LHL} \Rightarrow -1 = k$

$\Rightarrow$ $k = -1$ **(1)**

4. Find the value of k, so that the function f defined by

$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$.

HOTS; Delhi 2012C; Foreign 2011

A function $f(x)$ is said to be continuous at point $x = a$, if LHL at $(x = a)$ = RHL at $(x = a) = f(a)$. So, to find the value of k, we equate any one of LHL or RHL to $f(a)$ and simplify it.

Given function is $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

Also, given that function is continuous at $x = \pi/2$.

$\therefore$ At $x = \dfrac{\pi}{2}$, LHL = RHL = $f\left(\dfrac{\pi}{2}\right)$ ...(i) **(1)**

Now, LHL $= \lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{k\cos x}{\pi - 2x}$

$$\text{LHL} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{k\sin h}{\pi - \pi + 2h}$$

$\left[\text{put } x = \dfrac{\pi}{2} - h, \text{ when } x \to \dfrac{\pi}{2}, \text{ then } h \to 0\right]$

$\left[\because \cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta\right]$

$$= \lim_{h \to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$\Rightarrow$ LHL $= k/2$ **(1½)**

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3$$

(1/2)

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

Hence, $k = 6$ **(1)**

5. Find the value of a for which the function f is defined as $f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$

is continuous at $x = 0$. **Delhi 2011**

Given function is

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad \text{LHL} = \text{RHL} = f(0)$ ...(i) **(1)**

Now, $\text{LHL} = \lim_{x \to 0^-} a \sin \frac{\pi}{2}(x+1)$

$$\text{LHL} = \lim_{h \to 0} a \sin \frac{\pi}{2}(-h+1)$$

[put $x = 0 - h = -h$]

$$= \lim_{h \to 0} \frac{a \sin \left[\frac{\pi(-h+1)}{2}\right]}{\frac{\pi(-h+1)}{2}} \times \frac{\pi(-h+1)}{2}$$

$\left[\because \text{ multiplying numerator and denominator by } \frac{\pi(-h+1)}{2}\right]$

$$= \lim_{h \to 0} \frac{a\pi(-h+1)}{2} \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$\Rightarrow \quad \text{LHL} = \frac{a\pi}{2}$ [put $h = 0$] **(1)**

Now, $\text{RHL} = \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3}$

Put $x = 0 + h = h$, we get

$$\text{RHL} = \lim_{h \to 0} \frac{\tan h - \sin h}{h^3} = \lim_{h \to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$\text{RHL} = \lim_{h \to 0} \frac{\tan h - \sin h}{h^3} = \lim_{h \to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h - \sin h \cos h}{h^3 \cos h} = \lim_{h \to 0} \frac{\sin h (1 - \cos h)}{h^3 \cos h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1 - \cos h}{h^2} \cdot \lim_{h \to 0} \frac{1}{\cos h}$$

$$= 1 \times \lim_{h \to 0} \frac{1 - \cos h}{h^2} \times 1$$

$$\left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{1}{\cos h} = \frac{1}{\cos 0} = \frac{1}{1} = 1\right]$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h^2} \quad \left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2}\right]$$

$$= \lim_{h \to 0} \frac{2 \times \sin^2 \frac{h}{2}}{\left(\frac{h^2}{4} \times 4\right)} = \lim_{h \to 0} \frac{2}{4} \times \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4}}$$

[∵ multiplying numerator and denominator by 4]

$$= \frac{1}{2} \times \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 = \frac{1}{2} \times 1 \times 1$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \right]$$ **(1)**

$$\Rightarrow \text{RHL} = \frac{1}{2}$$

Now, from Eq. (i), we have LHL = RHL

$$\therefore \quad \frac{a\pi}{2} = \frac{1}{2} \Rightarrow a\pi = 1 \Rightarrow a = \frac{1}{\pi}$$ **(1)**

6. If the function $f(x)$ given by

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

is continuous at $x = 1$, then find the values of a and b. **Delhi 2011; All India 2010**

The given function is

$$f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$$

Given that $f(x)$ is continuous at $x = 1$.

$\therefore \quad \text{LHL} = \text{RHL} = f(x) \quad \ldots\text{(i)}$

Now, $\text{LHL} = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} (5ax - 2b)$ **(1)**

$\text{LHL} = \lim_{h \to 0} [5a(1-h) - 2b]$

[put $x = 1 - h$, when $x \to 1, h \to 0$]

$= \lim_{h \to 0} (5a - 5ah - 2b)$

$\Rightarrow \quad \text{LHL} = 5a - 2b \quad$ [put $h = 0$] **(1)**

Now, $\text{RHL} = \lim_{x \to 1^+} (3ax + b)$

$\text{RHL} = \lim_{h \to 0} [3a(1+h) + b]$

[put $x = 1 + h$, when $x \to 1, h \to 0$]

$= \lim_{h \to 0} (3a + 3ah + b)$

$\Rightarrow \quad \text{RHL} = 3a + b \quad$ [put $h = 0$]

Also, given that $f(1) = 11$ **(1)**

Now, from Eq. (i), we have

$$\text{RHL} = f(1)$$

$\Rightarrow \qquad 3a + b = 11 \qquad \dots(ii)$

and $\qquad \text{LHL} = f(1)$

$\Rightarrow \qquad 5a - 2b = 11 \qquad \dots(iii)$

On multiplying Eq. (ii) by 5 and Eq. (iii) by 3 and then subtracting, we get

$$15a + 5b = 55$$
$$15a - 6b = 33$$
$$- \quad + \quad -$$
$$11b = 22$$

$\Rightarrow \qquad b = 2$

On putting the value of b in Eq. (ii), we get

$$3a + 2 = 11 \Rightarrow 3a = 9 \Rightarrow a = 3$$

Hence, $a = 3$ and $b = 2$. **(1)**

7. Find the values of a and b such that the following function $f(x)$ is a continuous function.

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$

Delhi 2011

The given function is continuous that means $f(x)$ is continuous in its domain [2,10], so we take $x = 2$ and $x = 10$ to check continuity and then find a and b.

The given function is

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 2$ and at $x = 10$.

$\therefore$ By definition,

$$\text{LHL} = \text{RHL} = f(2) \quad \ldots(\text{i})$$

and $$\text{LHL} = \text{RHL} = f(10) \quad \ldots(\text{ii})$$ **(1)**

Now, first we calculate LHL and RHL at $x = 2$.

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5$$

$\Rightarrow$ $\text{LHL} = 5$

and $\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$

$$\text{RHL} = \lim_{h \to 0} a(2 + h) + b = \lim_{h \to 0} (2a + ah + b)$$

[put $x = 2 + h$, when $x \to 2, h \to 0$]

$= 2a + b$ [put $h = 0$]

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow$ $2a + b = 5$ ...(iii) **(1)**

Now, we find LHL and RHL at $x = 10$.

$$\text{LHL} = \lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$$

$$\text{LHL} = \lim_{h \to 0} [a(10 - h) + b]$$

[put $x = 10 - h$, when $x \to 10, h \to 0$]

$$= \lim_{h \to 0} (10a - ah + b)$$

$\Rightarrow$ LHL $= 10a + b$ [put $h = 0$]

and RHL $= \lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21 = 21$

Now, from Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow$ $10a + b = 21$ ...(iv) **(1)**

On subtracting Eq. (iv) from Eq. (iii), we get

$$-8a = -16 \Rightarrow a = 2$$

On putting $a = 2$ in Eq. (iv), we get

$$20 + b = 21$$

$\therefore$ $b = 1$

Hence, $a = 2$ and $b = 1$. **(1)**

8. Find the relationship between a and b, so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$. **All India 2011**

The given function is $f(x) = \begin{cases} ax+1, \text{ if } x \le 3 \\ bx+3, \text{ if } x > 3 \end{cases}$

Also, given that $f(x)$ is continuous at point $x = 3$.

$\therefore \quad \text{LHL} = \text{RHL} = f(3) \qquad \ldots(i)$ **(1)**

Now, $\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax+1)$

$\text{LHL} = \lim_{h \to 0} [a(3-h)+1]$

[put $x = 3-h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (3a - ah + 1)$

$\Rightarrow \quad \text{LHL} = 3a + 1 \qquad$ [put $h = 0$] **(1)**

$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx+3)$

$\text{RHL} = \lim_{h \to 0} [b(3+h)+3]$

[put $x = 3+h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (3b + bh + 3)$

$\Rightarrow \quad \text{RHL} = 3b + 3 \qquad$ [put $h = 0$] **(1)**

From Eq. (i), we have

$\text{LHL} = \text{RHL} \Rightarrow 3a + 1 = 3b + 3$

$\Rightarrow$ $3a - 3b = 2$, which is the required relation between a and b. **(1)**

9. Find the value of k, so that the function f defined by $f(x) = \begin{cases} kx+1, \text{ if } x \le \pi \\ \cos x, \text{ if } x > \pi \end{cases}$ is continuous at $x = \pi$. **Foreign 2011**

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

Also, given that $f(x)$ is continuous at $x = \pi$.

$\therefore \quad \text{LHL} = \text{RHL} = f(\pi) \qquad \ldots(i)$ **(1)**

Now, $\text{LHL} = \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} (kx + 1)$

$\text{LHL} = \lim_{h \to 0} [k(\pi - h) + 1]$

[put $x = \pi - h$, when $x \to \pi, h \to 0$]

$= \lim_{h \to 0} (k\pi - kh + 1)$

$= k\pi + 1$ [put $h = 0$] **(1)**

and $\text{RHL} = \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \cos x$

$\text{RHL} = \lim_{h \to 0} \cos(\pi + h)$

[put $x = \pi + h$, when $x \to \pi, h \to 0$]

$= \cos \pi$ [put $h = 0$] **(1)**

$= -1$ [$\because \cos \pi = -1$]

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \quad k\pi + 1 = -1$$

$$\Rightarrow \quad k\pi = -2$$

$$\therefore \quad k = \frac{-2}{\pi}$$ **(1)**

10. For what values of λ, is the function

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

is continuous at $x = 0$? **Foreign 2011**

The given function is

$$f(x)=\begin{cases}\lambda\,(x^2-2x)\,, & \text{if } x\le 0\\ 4x+1 & ,\text{if } x>0\end{cases}$$

Also, given that $f(x)$ is continuous at $x=0$.

$\therefore$ $\quad$ LHL = RHL = $f(0)$ ...(i)

Now, $\quad$ $\text{LHL}=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\lambda\,(x^2-2x)$ **(1)**

$\text{LHL}=\lim_{h\to 0}\lambda\,(h^2+2h)$

[put $x=0-h=-h$, when $x\to 0, h\to 0$]

$=\lambda\,(0)$ [put $h=0$] **(1)**

$=0$

and $\quad$ $\text{RHL}=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(4x+1)$

$\text{RHL}=\lim_{h\to 0}(4h+1)$

[put $x=0+h=h$, when $x\to 0, h\to 0$]

$\Rightarrow$ $\quad$ RHL = 1 [put $h=0$] **(1)**

Thus, $\quad$ LHL $\ne$ RHL

But it is given that LHL = RHL [from Eq. (i)]

Therefore, we get a contradiction.

Hence, there doesn't exist any real value of λ for which $f(x)$ is continuous at $x=0$. **(1)**

11. Discuss the continuity of the function $f(x)$ at $x=1/2$, when $f(x)$ is defined as follows:

$$f(x)=\begin{cases}1/2+x, & 0\le x<1/2\\ 1, & x=1/2\\ 3/2+x, & 1/2<x\le 1\end{cases}$$

Delhi 2011C

Here, we find LHL, RHL and $f\left(\frac{1}{2}\right)$. If LHL = RHL = $f\left(\frac{1}{2}\right)$, then we say that $f(x)$ is continuous at $x=\frac{1}{2}$, otherwise $f(x)$ is discontinuous at $x = \frac{1}{2}$.

The given function is

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$$

We have to check continuity of $f(x)$ at $x = \frac{1}{2}$.

(1)

Now, $\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} + x\right)$

$$\text{LHL} = \lim_{h \to 0} \left(\frac{1}{2} + \frac{1}{2} - h\right)$$

$$\left[\text{put } x = \frac{1}{2} - h, \text{ when } x \to \frac{1}{2}, h \to 0\right]$$

$$= \lim_{h \to 0} (1 - h)$$

$= 1$ [put $h = 0$] **(1)**

$$\text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} + x\right)$$

$$\text{RHL} = \lim_{h \to 0} \left(\frac{3}{2} + \frac{1}{2} + h\right)$$

$$\left[\text{put } x = \frac{1}{2} + h, \text{ when } x \to \frac{1}{2}, h \to 0\right]$$

$= \lim_{h \to 0} (2 + h) = 2$ [put $h = 0$] **(1)**

Now, we know that a function $f(x)$ is said to be continuous at point $x = a$ if continuous at point $x = a$, if

$$LHL = RHL = f(a).$$

Here, $LHL \neq RHL$ at $x = 1/2$.

Hence, $f(x)$ is discontinuous at $x = \frac{1}{2}$. **(1)**

12. Find the value of a, if the function $f(x)$ defined by

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

is continuous at $x = 2$.

All India 2011C; Delhi 2009C

Given, $f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$

is continuous at point $x = 2$.

$\therefore$ By definition of continuity of a function at a point, we have

$$LHL = RHL = f(2) \quad \text{...(i)}$$ **(1)**

Now, $LHL = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1)$

$\Rightarrow LHL = \lim_{h \to 0} [2(2 - h) - 1]$

[put $x = 2 - h$, when $x \to 2, h \to 0$]

$= \lim_{h \to 0} (4 - 2h - 1) = \lim_{h \to 0} (3 - 2h)$

$= 3$ [put $h = 0$] **(1½)**

Also, from the given function, we have

$$f(2) = a$$ **(1/2)**

On putting the values of $f(2)$ and LHL in Eq. (i), we get

$$3 = a$$

$\Rightarrow$ $a = 3$ **(1)**

13. Find the values of a and b such that the function defined as follows is continuous.

$$f(x) = \begin{cases} x+2, & x \le 2 \\ ax+b, & 2 < x < 5 \\ 3x-2, & x \ge 5 \end{cases}$$

Delhi 2010, 2010C

- Given function is $f(x) = \begin{cases} x+2, & x \le 2 \\ ax+b, & 2 < x < 5 \\ 3x-2, & x \ge 5 \end{cases}$

Also, given that $f(x)$ is continuous at $x = 2$ and $x = 5$.

$\therefore$ By definition of continuity, we get

$$\text{LHL} = \text{RHL} = f(2) \quad \ldots\text{(i)}$$

and $$\text{LHL} = \text{RHL} = f(5) \quad \ldots\text{(ii)}$$

First, we find LHL and RHL at $x = 2$. **(1)**

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2)$$

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} (2-h+2)$$

[put $x = 2-h$, when $x \to 2, h \to 0$]

$$= \lim_{h \to 0} (4-h)$$

$$\Rightarrow \quad \text{LHL} = 4 \quad \text{[put } h = 0\text{]}$$

Now, $\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax+b)$

$$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [a(2+h)+b]$$

[put $x = 2+h$, when $x \to 2, h \to 0$]

$$= \lim_{h \to 0} (2a + ah + b)$$

$$\Rightarrow \quad \text{RHL} = 2a + b \quad \text{[put } h = 0\text{]}$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$\therefore \quad 2a + b = 4$ ...(iii) **(1)**

Now, we find LHL and RHL at $x = 5$.

$$\text{LHL} = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (ax + b)$$

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} [a(5 - h) + b]$$

[put $x = 5 - h$, when $x \to 5, h \to 0$]

$$= \lim_{h \to 0} (5a - ah + b)$$

$\Rightarrow \quad \text{LHL} = 5a + b$ [put $h = 0$]

and $\quad \text{RHL} = \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 2)$

$$= \lim_{h \to 0} [3(5 + h) - 2]$$

[put $x = 5 + h$, when $x \to 5, h \to 0$]

$$= \lim_{h \to 0} (15 + 3h - 2)$$

$\Rightarrow \text{RHL} = 13$ [put $h = 0$]

$\therefore$ From Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow \quad 5a + b = 13$ ...(iv) **(1)**

On subtracting Eq. (iv) from Eq. (iii), we get

$$-3a = -9 \quad \Rightarrow \quad a = 3$$

Put $a = 3$ in Eq. (iv), we get

$$15 + b = 13 \quad \Rightarrow \quad b = -2$$

Hence, $\quad a = 3$ and $b = -2$ **(1)**

14. For what value of k, is the function defined by

$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \le 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$, continuous at $x = 0$?

Also, write whether the function is continuous at $x = 1$. **Delhi 2010, 2010C**

The given function is

$$f(x) = \begin{cases} k(x^2+2), & \text{if } x \le 0 \\ 3x+1, & \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad \text{LHL} = \text{RHL} = f(0) \quad \ldots\text{(i)}$ **(1/2)**

Now, $\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} k(x^2+2)$

$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} k(h^2+2)$

[put $x = 0 - h = -h$, when $x \to 0, h \to 0$]

$= \lim_{h \to 0} (kh^2 + 2k)$

$\Rightarrow \quad \text{LHL} = 2k$ [put $h = 0$] **(1/2)**

and $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3x+1)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} (3h+1)$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$\Rightarrow \quad \text{RHL} = 1$ [put $h = 0$] **(1/2)**

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL} \Rightarrow 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2} \quad \textbf{(1/2)}$$

Now, we check the continuity of the given function $f(x)$ at $x = 1$.

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x + 1)$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} [3(1 - h) + 1]$$

[put $x = 1 - h$, when $x \to 1, h \to 0$]

$$= \lim_{h \to 0} (3 - 3h + 1)$$

$$= 4 \quad \text{[put } h = 0] \ \textbf{(1/2)}$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x + 1)$$

$$\Rightarrow \text{RHL} = \lim_{h \to 0} [3(1 + h) + 1]$$

[put $x = 1 + h$, when $x \to 1, h \to 0$]

$$= \lim_{h \to 0} [3 + 3h + 1]$$

$$= 4 \quad \text{[put } h = 0] \ \textbf{(1/2)}$$

and $f(1) = 4$ from given function.

$\because$ At $x = 1$, LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$. **(1)**

15. Find all points of discontinuity of f, where f is defined as follows:

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

Delhi 2010

Firstly, verify continuity of the given function at $x=-3$ and $x=3$. Then, point at which the given function is discontinuous will be the point of discontinuity.

The given function is $f(x)=\begin{cases}|x|+3, & x\leq -3\\ -2x, & -3<x<3\\ 6x+2, & x\geq 3\end{cases}$

First, we verify continuity at $x=-3$ and then at $x=3$.

Continuity at $x=-3$

$\text{LHL}=\lim_{x\to(-3)^-} f(x)=\lim_{x\to(-3)^-}(|x|+3)$

$\Rightarrow \text{LHL}=\lim_{h\to 0}(|-3-h|+3)$

[put $x=-3-h$, when $x\to -3$, $h\to 0$]

$=|-3|+3$ [put $h=0$]

$=3+3$ [$\because |-x|=x, \forall x\in R$]

$=6$ **(1/2)**

$\text{RHL}=\lim_{x\to(-3)^+} f(x)=\lim_{x\to(-3)^+}(-2x)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [-2(-3+h)]$

[put $x = -3 + h$, when $x \to -3, h \to 0$]

$= \lim_{h \to 0} (6 - 2h)$

$\Rightarrow \quad \text{RHL} = 6$ [put $h = 0$] **(1/2)**

Also, $f(-3)$ = value of $f(x)$ at $x = -3$

$= |-3| + 3$

$= 3 + 3 = 6$ [$\because |-x| = x, \forall\, x \in R$] **(1/2)**

$\because \quad \text{LHL} = \text{RHL} = f(-3)$

$\therefore$ $f(x)$ is continuous at $x = -3$. So, $x = -3$ is the point of continuity. **(1/2)**

Continuity at $x = 3$

$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -2x$

$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} -2(3 - h)$

[put $x = 3 + h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (-6 + 2h)$

$\Rightarrow \quad \text{LHL} = -6$ [put $h = 0$] **(1/2)**

and $\quad \text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [6(3 + h) + 2]$

[put $x = 3 - h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (18 + 6h + 2)$

$\Rightarrow \quad \text{RHL} = 20$ [put $h = 0$] **(1/2)**

$\because \quad \text{LHL} \neq \text{RHL}$

As $f(x)$ is a polynomial function in a given interval, so it is continuous in a given interval but $f(x)$ is not continuous at $x = 3$. So, $x = 3$ is the point of discontinuity of $f(x)$. **(1)**

16. Show that the function $f(x)$ defined by

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x\ , & x>0\\ 2\ , & x=0\\ \dfrac{4(1-\sqrt{1-x})}{x}\ , & x<0\end{cases}$$

is continuous at $x=0$. **All India 2009C**

To show that the given function is continuous at $x=0$, we show that

$$(\text{LHL})_{x=0}=(\text{RHL})_{x=0}=f(0) \qquad \ldots\text{(i)}\ \mathbf{(1)}$$

Now, given function is

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & x>0\\ 2, & x=0\\ \dfrac{4(1-\sqrt{1-x})}{x}, & x<0\end{cases}$$

$$\text{LHL}=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{4(1-\sqrt{1-x})}{x}$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} \frac{4[1 - \sqrt{1-(0-h)}]}{0-h}$$

[put $x = 0 - h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h}]}{-h}$$

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h}]}{-h} \times \frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}$$

$$= \lim_{h \to 0} \frac{4[(1)^2 - (\sqrt{1+h})^2]}{-h[1+\sqrt{1+h}]}$$

[$\because (a-b)(a+b) = a^2 - b^2$]

$$= \lim_{h \to 0} \frac{4[1-(1+h)]}{-h[1+\sqrt{1+h}]}$$

$$= \lim_{h \to 0} \frac{-h \times 4}{-h[1+\sqrt{1+h}]} = \lim_{h \to 0} \frac{4}{1+\sqrt{1+h}}$$

$$= \frac{4}{1+\sqrt{1}} = \frac{4}{2} = 2 \qquad \text{[put } h = 0]$$

$\Rightarrow$ LHL = 2 **(1)**

Now, $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sin x}{x} + \cos x \right)$

$$\Rightarrow \text{RHL} = \lim_{h \to 0} \left(\frac{\sin h}{h} + \cos h \right)$$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{\sin h}{h} + \lim_{h \to 0} \cos h \qquad \text{[put } h = 0]$$

$$= 1 + \cos 0° \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1 + 1 \qquad [\because \cos 0 = 1]$$

$$= 2$$ **(1)**

Also, given that at $x = 0$, $f(x) = 2 \Rightarrow f(0) = 2$

Since, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) = 2$

Hence, $f(x)$ is continuous at $x = 0$. **(1)**

17. For what value of k, is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x+1 & , x<2 \\ k & , x=2 \\ 3x-1 & , x>2 \end{cases}$$

Delhi 2008

Do same as Que 12. **[Ans.** $k = 5$**]**

18. If $f(x)$ defined by the following, is continuous at $x = 0$, then find the values of a, b and c.

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c & , \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , \text{if } x>0 \end{cases}$$

HOTS; All India 2008

Given function is

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c & , \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , \text{if } x>0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $\quad \text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} \frac{\sin[-(a+1)h] + \sin(-h)}{-h}$$

[put $x = 0 - h = -h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{-\sin(a+1)h - \sin h}{-h}$$

[$\because \sin(-\theta) = -\sin\theta$]

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h}$$

$$= \lim_{h \to 0} \frac{2 \sin\left[\frac{(a+1)h + h}{2}\right] \cos\left[\frac{(a+1)h - h}{2}\right]}{h}$$

$$\left[\because \sin C + \sin D = 2 \sin \frac{(C+D)}{2} \cdot \cos \frac{(C-D)}{2}\right]$$

$$= \lim_{h \to 0} \frac{2 \sin\left[\frac{(a+1)h + h}{2}\right] \cos\left[\frac{(a+1)h - h}{2}\right]}{\left[\frac{(a+1)h + h}{2}\right] \times h} \times \frac{(a+1)h + h}{2}$$

$$\left[\text{multiplying and dividing by } \frac{(a+1)h + h}{2}\right]$$

$$= \lim_{h \to 0} \frac{2 \cos\left[\frac{(a+1)h - h}{2}\right]}{h} \times \frac{h[a+1+1]}{2}$$

$$\left[\begin{array}{l} \because \lim_{x \to 0} \frac{\sin x}{x} = 1 \\ \therefore \lim_{h \to 0} \frac{\sin\frac{[(a+1)h + h]}{2}}{\frac{(a+1)h + h}{2}} = 1 \end{array}\right]$$

$$= \lim_{h \to 0} \frac{2(a+2)}{2} \cos\left[\frac{(a+1)h - h}{2}\right]$$

$= (a+2) \cos 0°$ [put $h = 0$]

$= (a+2) \times 1$ [$\because \cos 0° = 1$]

$= a + 2$ **(1½)**

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}$$

$$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} \frac{\sqrt{h + bh^2} - \sqrt{h}}{bh^{3/2}}$$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{\sqrt{h(1+bh)} - \sqrt{h}}{bh^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,[\sqrt{1+bh} - 1]}{b\sqrt{h} \cdot h} \quad [\because h^{3/2} = \sqrt{h} \cdot h]$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh} \times \frac{\sqrt{1+bh} + 1}{\sqrt{1+bh} + 1}$$

[multiplying and dividing by $\sqrt{1+bh} + 1$]

$$= \lim_{h \to 0} \frac{(\sqrt{1+bh})^2 - (1)^2}{bh[\sqrt{1+bh} + 1]}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \lim_{h \to 0} \frac{1 + bh - 1}{bh[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \to 0} \frac{bh}{bh[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+bh} + 1}$$ **(1½)**

$\Rightarrow \text{RHL} = \frac{1}{1+1} = \frac{1}{2}$ [put $h = 0$]

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \quad a + 2 = \frac{1}{2}$$

$$\Rightarrow \quad a = \frac{1}{2} - 2 \Rightarrow a = -\frac{3}{2}$$

Also, given that $f(0) = c$

Again from Eq. (i), we have

$$\text{RHL} = f(0)$$

$$\Rightarrow \quad c = 1/2$$

Hence, we get $a = -\frac{3}{2}$, $c = \frac{1}{2}$ and b may take any real value. **(1)**

NOTE Here, we cannot find any real and unique value of b that means b may take any real value, i.e. $b \in R$.

19. Show that $f(x) = \begin{cases} 5x-4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at $x = 1$. **Delhi 2008C**

Given function is

$$f(x) = \begin{cases} 5x-4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$$

To show that $f(x)$ is continuous at $x = 1$, we need to prove

$$LHL_{x=1} = RHL_{x=1} = f(1) \quad \text{...(i)} \quad \textbf{(1)}$$

Now, $LHL = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x - 4)$

$\Rightarrow \quad LHL = \lim_{h \to 0} [5(1-h) - 4]$

[put $x = 1 - h$, when $x \to 1$, $h \to 0$]

$= \lim_{h \to 0} (5 - 5h - 4) = \lim_{h \to 0} (1 - 5h)$

$LHL = 1$ [put $h = 0$] **(1)**

$RHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^3 - 3x)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [4(1+h)^3 - 3(1+h)]$

[put $x = 1 + h$, when $x \to 1, h \to 0$]

$= 4(1)^3 - 3(1)$ [put $h = 0$]

$\Rightarrow \quad \text{RHL} = 4 - 3 = 1$ **(1)**

Also, from given function,

$f(1) =$ Value of $f(x)$ at $x = 1$

$\Rightarrow \quad f(1) = 5(1) - 4$

[put $x = 1$ in $f(x) = 5x - 4$]

$= 5 - 4 = 1$

Here, $(\text{LHL})_{x=1} = (\text{RHL})_{x=1} = f(1)$ **(1)**

Hence, $f(1)$ is continuous at $x = 1$.

20. If the following function $f(x)$ is continuous at $x = 0$, then find the value of k.

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}.$$

All India 2008C

Do same as Que 1. **[Ans. $k = 1$]**